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Existence Uniqueness Continuation and Stability of Variable Order Caputo Type Fractional Differential Equations with Delay

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KEYWORDS

Variable Order Fractional Delay Differential Equation (VOFDDE), Existence–Uniqueness, Continuation theorem, Ulam–Hyers stability.

ABSTRACT:
In this paper, we discuss about the existence, uniqueness and continuation of Caputo type Variable Order Fractional Delay Differential Equation (VOFDDE)

{█(〖(_C^)D〗 _(0,t)^s(t) y(t)=f(t,y(t),y(t-τ)),0<s(t)<1@y(t)=y_0   at t=0,y∈R,t∈(0,∞))┤

We extend the continuation theorem for the variable order fractional delay differential equation and then we present different types of Ulam–Hyers stability results for the Caputo type VOFDDE

1. Introduction

Because of the wide applications in the fields of science and engineering [1, 12, 13, 15] fractional calculus has broadened over the past few years. In recent times, the study about the Fractional differential equation has much interest [16, 17, and 18]. Variable-order fractional differential equations with delay are a relatively less known branch of mathematics that offers remarkable opportunities to simulate interdisciplinary processes. The existence and uniqueness of the differential equation plays a vital role in the theory of differential equations. In this paper we discuss about the existence, uniqueness and stability of variable order Caputo type Fractional Differential Equations with Delay. In variable order, the order can vary continuously as a function of dependent or independent variable.

Consider the Caputo type VOFDDE initial value problem as

$$\begin{cases} {}_{C}D_{0,t}^{s(t)} y(t) = f(t, y(t), y(t-\tau)), 0 < s(t) < 1 \\ \quad y(t) = y_0 \ \ at\, t = 0, y \in \mathbb{R},\ t \in (0, \infty) \end{cases} \tag{1.1}$$

Where ${}_{C}D_{0,t}^{s(t)}$ (.) is the Caputo derivative with the variable order s(t) defined in (2.3).

2. Preliminaries

In this paper, we focus on the variable order Caputo derivative. We obtained the fractional derivative and integral with the variable-order with delay by extending the fractional derivative and integral of the constant order [1, 2, 3, 4, and 14].

Definition 1: [5]

The variable order Riemann–Liouville integral of function $f(u)$ is

$${}_{RL}D_{0,t}^{-s(t)} f(u) = \frac{1}{\Gamma(s(t))} \int_0^t (u-\lambda)^{s(t)-1} f(\lambda) d\lambda, \quad t > 0, s(t) > 0 \tag{2.1}$$

Definition 2: [5]

The variable order Riemann– Liouville derivative of function f (u) is defined as

$${}_{RL}D_{0,t}^{s(t)} f(u) = \frac{1}{\Gamma(n-s(t))} \frac{d^n}{dt^n} \int_0^t (u-\lambda)^{n-s(t)-1} f(\lambda) d\lambda,\ \ t > 0, s(t) > 0 \tag{2.2}$$

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Definition 3: [5]
The variable order Caputo derivative of $f\ (u)$ is
$${}_{C}D_{0,t}^{s(t)}f(u) = \frac{1}{\Gamma(n-s(t))}\int_0^t (u-\lambda)^{s(t)-1} f^{(n)}(\lambda)d\lambda,\ \ t>0, s(t)>0 \tag{2.3}$$

Definition 4: [5]
The derivatives (2.2) and (2.3) are not often equivalent; however, they can be linked by the following relationship [6]
$${}_{RL}D_{0,t}^{s(t)}f(t) = \sum_{k=0}^{n-1}\frac{f^{(k)}(0)t^{k-s(t)}}{\Gamma(1+k-s(t))} = {}_{C}D_{0,t}^{s(t)}f(t) \tag{2.4}$$
When $0 < s(t) < 1$, then the relation between (2.2) and (2.3) can be defined as
$${}_{C}D_{0,t}^{s(t)}f(t) = {}_{RL}D_{0,t}^{s(t)}\{f(t)-f(0)\} \tag{2.5}$$

Lemma 1: [2, 3, 6].
We assume that $f\ (x,t)$ is a continuous function. Then the second kind of nonlinear Volterra Type equation here.integral equation is equivalent to variable order initial problem (1.1) as
$$y(t) = y_0 + \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} f(\lambda, y(\lambda), y(\lambda-\tau))d\lambda \tag{2.6}$$
then every solution of (2.6) is also the solution of (1.1) and vice versa.
Proof:
By applying the operator ${}_{RL}D_{0,t}^{-s(t)}f(u)$ to both sides of (1.1), and using (2.5) and the initial condition $y(t) = y_0\ at\ t = 0$,
$${}_{RL}D_{0,t}^{-s(t)}\ {}_{C}D_{0,t}^{s(t)}y(t) = {}_{RL}D_{0,t}^{-s(t)}f(t, y(t), y(t-\tau))$$
$$y(t) - y_0 = \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} f(\lambda, y(\lambda), y(\lambda-\tau))d\lambda$$
$$y(t) = y_0 + \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} f(\lambda, y(\lambda), y(\lambda-\tau))d\lambda$$
We can reduce the problem (1.1) into the equivalent Volterra nonlinear integral Equation (2.6). The proof is complete.
Lemma 2: [5, 7, 8].
Assume that $S \subseteq C[0,T]$. Then S is called pre-compact if $\{y(t) : y \in M\}$ is uniformly bounded and equicontinuous on [0, T].
Lemma 3: [5, 7, 8].
Suppose that X is a Banach space, and $S \subset X$, where S is the closed bounded convex set, and assume that $T : S \to S$ is the continuous completely. Then there exists a fixed point of T in S.
Lemma 4: [5, 7, 8].
Suppose that a non-empty closed set S is a subset of a Banach space X, and assume that $a_n \geq 0$, then $\sum_{n=0}^{\infty} a_n$ converges for all $n \in$ N. Further if we assume $A: S \to S$ satisfies
$\|P^n y_1 - P^n y_2\| \leq a_n \|y_1 - y_2\|, y_1, y_2 \in S$.
Then, for any $u^* \in S$ is defined the unique fixed point of P.

3. Theorems on Existence, Uniqueness, and Continuation
First, we prove the local existence and uniqueness of the solution of (1.1) by using the following hypothesis.

Hypothesis 1 (H1):
Assume that $f : [0, +\infty) \times R \to R$ in (1.1) is a continuous function. Then the function f fulfills the Lipschitz's condition,
i.e., $|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$, where $L > 0$.

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Hypothesis 2 (H2):
Assume that the function f have weak singularity, with respect to t then there exists a constant $\eta \in (0,1]$ such that $(\mathfrak{J}y)(t) = t^{\eta} f(t, y(t), y(t-\tau))$ is a continuous bounded map defined on $[0,T] \times [0,T], and\ T > 0$.

Theorem 1:
Assume that conditions (H1) and (H2) are hold. Then (1.1) has at least one solution and $y \in C[0,h^*], for\ some\ h^* \in (0,T]$

Proof

Let $\Omega = \{y \in C[0,T]: \|y - y_0\|_{C[0,T]} = \sup_{t\in[0,T]} |\, y - y_0 \,| \le \varphi\}$

Where $\varphi > 0$. Because $\mathfrak{J}$ is bounded, so there exists a constant $N > 0$, such that
$\sup\{|\,(\mathfrak{J}y)(t)\,|: t \in [0,T], u \in \Omega\} \le N$.

If we take

$$\Lambda_{h^*} = \left\{y: y \in C[0,h^*], \sup_{t\in[0,T]} |\, y - y_0 \,| \le \varphi\right\},$$

where $h^* = \min\left\{\left(\frac{\varphi\Gamma(s(t)+1-\eta)}{N\Gamma(1-\eta)}\right)^{\frac{1}{s(t)-\eta}}, T\right\}, s(t) > \eta$.

It is trivial that $\Lambda_{h^*} \subseteq C[0,h^*]$ is bounded, closed, nonempty, and a convex subset.

And also $h^* \le T$.

Define $\aleph$ as

$$(\aleph y)(\mathrm{t}) = y_0 + \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} f(\lambda, y(\lambda), y(\lambda-\tau))d\lambda,\ t \in [0,h^*] \quad (3.1)$$

This gives

$$|(\aleph y)(\mathrm{t}) - y_0| \le \frac{N}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1}\lambda^{-\eta}\, d\lambda \le \frac{N\Gamma((1-\eta)}{\Gamma(s(t)+1-\eta)} h^{s(t)-\eta}$$

$$\le \frac{N\Gamma((1-\eta)}{\Gamma(s(t)+1-\eta)} \cdot \frac{\varphi\Gamma(s(t)+1-\eta)}{N\Gamma(1-\eta)} = \varphi$$

Thus

$|(\aleph y)(\mathrm{t}) - y_0| \le \varphi, for\ any\ y \in C[0,h^*]$,

$which\ shows\ that\ \aleph\Lambda_h \subset \Lambda_h$.

To prove the continuity of operator $\aleph$, we proceed as follows.

Let $y_n, y \in \Lambda_{h^*}$ such that $\|y_n - y\|_{C[0,h^*]} \to 0$ as $n \to \infty$,

since the operator $\mathfrak{J}$ is continuous we have $\|\mathfrak{J}y_n - \mathfrak{J}y\|_{[0,h^*]} \to 0$ as $n \to \infty$.

Now

$$\|(\aleph y_n)(t) - (\aleph y)(t)\|$$

$$= \left|\frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} f(\lambda, y_n(\lambda), y_n(\lambda-\tau))d\lambda - \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} f(\lambda, y(\lambda), y(\lambda-\tau))d\lambda\right|$$

$$\le \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1}\, |\, f(\lambda, y_n(\lambda), y_n(\lambda-\tau)) - f(\lambda, y(\lambda), y(\lambda-\tau))\,|\, d\lambda$$

$$\le \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1}\lambda^{-\eta}\, |\,(\mathfrak{J}y_n)\lambda - (\mathfrak{J}y)\lambda\,|\, d\lambda$$

$$\le \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1}\lambda^{-\eta} \|(\mathfrak{J}y_n)\lambda - (\mathfrak{J}y)\lambda\|_{[0,h^*]}\, d\lambda$$

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Thus,

$$\left\|(\aleph y_n)(t)-(\aleph y)(t)\right\|_{[0,h^*]} \le \frac{\Gamma((1-s(t))}{\Gamma(s(t)+1-\eta)} h^{s(t)-1} \left\|(\Im y_n)\lambda-(\Im y)\lambda\right\|_{[0,h^*]}$$

$$\left\|(\aleph y_n)(t)-(\aleph y)(t)\right\|_{[0,h^*]} \to 0 \ as\ n\to\infty.$$

Hence $\aleph$ is continuous.

Next we prove the continuity of $\aleph\Lambda_h$ as follows.

For this we let $y \in \Lambda_{h^*}\ and\ t_1,t_2 \in [0,h^*], t_1 \le t_2\ and for\ any\ \varepsilon>0,$

$$\frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1}\lambda^{-\eta}\,d\lambda = \frac{\Gamma(1-\eta)}{\Gamma(s(t)+1-\lambda)} t^{s(t)-\eta} \to 0, as\ t\to 0^+, where\ \eta\in[0,1).$$

Then there exists a $\bar{\eta}>0$, such that

$$\frac{2N}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1}\lambda^{-\eta}\,d\lambda < \varepsilon, \qquad t\in[0,h^*]$$

holds, so that for $t_1,t_2\in[0,\bar{\eta}]$ we have

$$\left|\frac{1}{\Gamma(s(t))}\int_0^{t_1}(t_1-\lambda)^{s(t)-1} f(\lambda,y(\lambda),y(\lambda-\tau))\,d\lambda - \frac{1}{\Gamma(s(t))}\int_0^{t_2}(t_2-\lambda)^{s(t)-1} f(\lambda,y(\lambda),y(\lambda-\tau))\,d\lambda\right|$$

$$\le \frac{N}{\Gamma(s(t))}\int_0^{t_1}(t_1-\lambda)^{s(t)-1}\lambda^{-\eta}\,d\lambda + \frac{N}{\Gamma(s(t))}\int_0^{t_2}(t_2-\lambda)^{s(t)-1}\lambda^{-\eta}\,d\lambda$$

$$<\varepsilon \tag{3.2}$$

For $\frac{\bar{\eta}}{2}\le t_1\le t_2\le h^*$

$$|(\aleph y)(t_1)-(\aleph y)(t_2)|$$

$$= \left|\frac{1}{\Gamma(s(t))}\int_0^{t_1}(t_1-\lambda)^{s(t)-1} f\big(\lambda,y(\lambda),y(\lambda-\tau)\big)\,d\lambda - \frac{1}{\Gamma(s(t))}\int_0^{t_2}(t_2-\lambda)^{s(t)-1} f\big(\lambda,y(\lambda),y(\lambda-\tau)\big)\,d\lambda\right|$$

Adding and subtracting $\frac{1}{\Gamma(s(t))}\int_0^{t_1}(t_2-\lambda)^{s(t)-1} f\big(\lambda,y(\lambda),y(\lambda-\tau)\big)\,d\lambda$, then simplifying the above equation, we get

$$|(\aleph y)(t_1)-(\aleph y)(t_2)| \le \left|\frac{1}{\Gamma(s(t))}\int_0^{t_1}[(t_1-\lambda)^{s(t)-1}-(t_2-\lambda)^{s(t)-1}] f\big(\lambda,y(\lambda),y(\lambda-\tau)\big)\,d\lambda\right|$$

$$+\left|\frac{1}{\Gamma(s(t))}\int_{t_1}^{t_2}(t_2-\lambda)^{s(t)-1} f\big(\lambda,y(\lambda),y(\lambda-\tau)\big)\,d\lambda\right| \tag{3.3}$$

The first term of (3.3) can be expressed as

$$\left|\frac{1}{\Gamma(s(t))}\int_0^{t_1}[(t_1-\lambda)^{s(t)-1}-(t_2-\lambda)^{s(t)-1}] f\big(\lambda,y(\lambda),y(\lambda-\tau)\big)\,d\lambda\right|$$

$$\le \frac{N}{\Gamma(s(t))}\int_0^{t_1}[(t_1-\lambda)^{s(t)-1}-(t_2-\lambda)^{s(t)-1}]\ |\ \lambda^{-\eta}\ |\,d\lambda$$

$$\le \frac{N}{\Gamma(s(t))}\int_0^{\frac{\bar{\eta}}{2}}[(t_1-\lambda)^{s(t)-1}-(t_2-\lambda)^{s(t)-1}]\ |\ \lambda^{-\eta}\ |\,d\lambda + \frac{N\left(\frac{\bar{\eta}}{2}\right)-\eta}{\Gamma(s(t))}\int_{\frac{\bar{\eta}}{2}}^{t_1}|\ [(t_1-\lambda)^{s(t)-1}-(t_2-\lambda)^{s(t)-1}]\ |\,d\lambda$$

$$\le \frac{2N}{\Gamma(s(t))}\int_0^{\frac{\eta_1}{2}}\left(\frac{\delta}{2}-\lambda\right)^{s(t)-1}\lambda^{-\eta}\,d\lambda + \frac{N\left(\frac{\bar{\eta}}{2}\right)-\eta}{\Gamma(s(t))}\left[\frac{(t_1-\lambda)^{s(t)}}{-1}-\frac{(t_2-\lambda)^{s(t)}}{-1}\right]\Big|_{\frac{\bar{\eta}}{2}}^{t_1}$$

$$\le \frac{2N}{\Gamma(s(t))}\int_0^{\frac{\eta_1}{2}}\left(\frac{\bar{\delta}}{2}-\lambda\right)^{s(t)-1}\lambda^{-\eta}\,d\lambda$$

$$+\frac{N\left(\frac{\overline{\eta}}{2}\right)-\eta}{\Gamma\left(s(t)\right)}[0+(t_2-t_1)^{s(t)}+\left(t_1-\frac{\overline{\eta}}{2}\right)^{s(t)}-\left(t_2-\frac{\overline{\eta}}{2}\right)^{s(t)}]$$

$$\le \varepsilon+\frac{N\left(\frac{\overline{\eta}}{2}\right)-\eta}{\Gamma\left(s(t)\right)}[(t_2-t_1)^{s(t)}+\left(t_1-\frac{\overline{\eta}}{2}\right)^{s(t)}-\left(t_2-\frac{\overline{\eta}}{2}\right)^{s(t)}]$$

The second term of (3.3) can be expressed as

$$\left|\frac{1}{\Gamma(s(t))}\int_{t_1}^{t_2}(t_2-\lambda)^{s(t)-1}f\left(\lambda,y(\lambda),y(\lambda-\tau)\right)d\lambda\right|\le\frac{N\left(\frac{\eta_1}{2}\right)-\eta}{\Gamma(s(t))}\int_{t_1}^{t_2}(t_2-\lambda)^{s(t)-1}d\lambda$$

$$\le\frac{N\left(\frac{\eta_1}{2}\right)-\eta}{\Gamma(s(t))}(t_2-t_1)^{s(t)} \tag{3.4}$$

Hence there exists

$\frac{\overline{\eta}}{2}>\overline{\eta_1}>0\ for\ \frac{\overline{\eta}}{2}\le t_1\le t_2\le h^*\ and\ \mid t_1-t_2\mid<\overline{\eta_1}\ such\ that$

$$\mid(\aleph y)(t_1)-(\aleph y)(t_2)\mid<2\varepsilon \tag{3.5}$$

Conditions (3.2) and (3.5) implies that $\{(\aleph y)(t): y\in h^*\}$ is equicontinuous and uniformly bounded since $\aleph\Lambda_{h^*}\subset\Lambda_{h^*}$. Therefore $\aleph\Lambda_{h^*}$ is pre-compact and hence the operator $\aleph$ is completely continuous. It gives the local existence of (1.1) by using Lemma 2 and 3.

Theorem 2:

Assume (H1) and (H2) are fulfilled. Then there exists the unique solution of IVP (1.1) for $u\in C[0,h^*]$, where $h^*\in(0,T]$.

Proof: Using Lemma 1, (1.1) and (2.6) are equivalent. It remains to prove only that (2.6) has one solution only. First, we have a non-empty and closed subset of the Banach space in the form

$$\Lambda_{h^*}=\left\{y: y\in C[0,h^*],\ \sup_{t\in[0,T]}\mid y-y_0\mid\le\varphi\right\},$$

We use the operator $\aleph$ as

$$(\aleph y)(t)=y_0+\frac{1}{\Gamma(s(t))}\int_0^t(t-\lambda)^{s(t)-1}f(\lambda,y(\lambda),y(\lambda-\tau))d\lambda,\ t\in[0,h^*].$$

we obtain the fixed point problem from the uniqueness of the solution to (2.6),

i.e., $y(t)=(\aleph y)(t)$.

So, it remains to prove that $\aleph$ has a unique fixed point.

We have

$$\left|(\aleph y)(t)-y_0\right|\le\frac{1}{\Gamma\left(s(t)\right)}\int_0^t(t-\lambda)^{s(t)-1}|f(\lambda,y(\lambda),y(\lambda-\tau)|\,d\lambda$$

$$\le\frac{\|f\|_{C[0,h^*]}\Gamma(1-s(t))}{\Gamma(1-\eta)}\frac{\varphi\Gamma(s(t)+1-\eta)}{\|f\|_{C[0,h^*]}\Gamma(1-s(t))}=\varphi, for\ any\ y\in\Lambda_{h^*}$$

Hence $\aleph y\in\Lambda_{h^*}, if\ y\in\Lambda_{h^*}$

$$|(\aleph y)(t_1)-(\aleph y)(t_2)|$$

$$=\left|\frac{1}{\Gamma\left(s(t)\right)}\int_0^{t_1}(t_1-\lambda)^{s(t)-1}f\left(\lambda,y(\lambda),y(\lambda-\tau)\right)d\lambda-\frac{1}{\Gamma\left(s(t)\right)}\int_0^{t_2}(t_2-\lambda)^{s(t)-1}f\left(\lambda,y(\lambda),y(\lambda-\tau)\right)d\lambda\right|$$

$$\le\frac{1}{\Gamma\left(s(t)\right)}\int_0^{t_1}[(t_1-\lambda)^{s(t)-1}-(t_2-\lambda)^{s(t)-1}]\left|f\left(\lambda,y(\lambda),y(\lambda-\tau)\right)\right|d\lambda+\frac{1}{\Gamma\left(s(t)\right)}\int_{t_1}^{t_2}(t_2-\lambda)^{s(t)-1}\left|f\left(\lambda,y(\lambda),y(\lambda-\tau)\right)\right|d\lambda$$

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$\leq \frac{\|f\|_{C[0,h^*]}}{\Gamma(s(t))}\int_0^{t_1}[(t_1-\lambda)^{s(t)-1}-(t_2-\lambda)^{s(t)-1}]\,d\lambda+\frac{\|f\|_{C[0,h^*]}}{\Gamma(s(t))}\int_{t_1}^{t_2}(t_2-\lambda)^{s(t)-1}\,d\lambda$

On integrating we obtain

$\left|(\aleph y)(t_1)-(\aleph y)(t_2)\right| \leq \frac{\|f\|_{C[0,h^*]}}{\Gamma(1+s(t))}[(t_2-t_1)^{s(t)}+(t_1)^{s(t)}-(t_2)^{s(t)}]$

This shows that $\aleph y$ is continuous.

Also, we have

$\left\|\aleph^n y-\aleph^n \bar{y}\right\|_{C[0,t]} \leq \frac{(L\delta^{s(t)})^n}{\Gamma(1+ns(t))}\|y-\bar{y}\|_{C[0,t]}, for\ every\ n \in \mathbb{N}\ and\ t \in [0,h^*].$ (3.6)

For $n = 0$, (3.6) is true.

By the fundamental concept of induction, the case $n-1$ is also true. We get

$\left\|\aleph^n y-\aleph^n \bar{y}\right\|_{C[0,t]} = \left\|\aleph(\aleph^{n-1}y)-\aleph(\aleph^{n-1}\bar{y})\right\|_{C[0,t]}$

$\frac{1}{\Gamma(s(t))}\max_{0\in[\delta,t]}\left|\int_0^{\delta}(\delta-\lambda)^{s(t)-1}\left[f\left(\lambda,\aleph^{n-1}y(\lambda),\aleph^{n-1}y(\lambda-\tau)\right)-f(\lambda,\aleph^{n-1}\bar{y}(\lambda),\aleph^{n-1}\bar{y}(\lambda-\tau))\right]d\,\lambda\right|$

The result (3.6) is obvious by the Lipschitz's condition and the induction hypothesis.

Also,

$\sum_{n=0}^{\infty}\frac{(L\delta^{s(t)})^n}{\Gamma(1+ns(t))}=E_{s(t)}(L\delta^{s(t)}),$

Here $E_s(.)$ is the Mittag-Leffler function, defined as $E_s(z)=\sum_{n=0}^{\infty}\frac{z^k}{1+sk}$.

Hence, we can apply Lemma 4 and deduce the uniqueness of equation (1.1).

Theorem 3:

Assume $(H1)$ and $(H2)$ are hold, then $y = y(t), t \in (0,\zeta)$ is non-continuable if only for some $\varsigma \in (0,\zeta/2)$ and any bounded closed subset $X \subset [\varsigma,+\infty) \times R$ then there exists $t^* \in [\varsigma,\zeta)$, such that $\left(t^*,y(t^*),y(t^*-\tau)\right) \notin X$.

Proof:

We explain the proof in two steps.

Let there exists $X \subset [\varsigma,,+\infty)\ X\ R$ and $\left\{\left(t,y(t),y(t-\tau)\right): t \in [\varsigma,\zeta)\right\} \subset X$.

The compactness of $X \Rightarrow \zeta < +\infty$. By (H1) there exists a positive K such that

$\sup_{(t,y)\in X}|f| \leq K.$

Step 1: To prove $\lim_{t\to\zeta^-} y(t)$ exists.

Let $G(t)=\int_0^{\varsigma}(t-\lambda)^{s(t)-1}\lambda^{-\eta}\,d\lambda, \quad t \in [2\varsigma,\zeta]$

$G(t)$ is uniformly continuous on $[2\varsigma,\zeta]$. For all $t_1,t_2 \in [2\varsigma,\zeta]$ $with$ $t_1 \leq t_2$ we have

$$|y(t_1)-y(t_2)| = \left|\frac{1}{\Gamma(s(t))}\int_0^{t_1}(t_1-\lambda)^{s(t)-1}f\left(\lambda,y(\lambda),y(\lambda-\tau)\right)d\lambda - \frac{1}{\Gamma(s(t))}\int_0^{t_2}(t_2-\lambda)^{s(t)-1}f\left(\lambda,y(\lambda),y(\lambda-\tau)\right)d\lambda\right|$$

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$$\leq \left|\frac{1}{\Gamma(s(t))}\int_0^{\varsigma}[(t_1-\lambda)^{s(t)-1}-(t_2-\lambda)^{s(t)-1}]\lambda^{-\eta}\,(\mathfrak{z}y)(\lambda)d\lambda\right|$$
$$+\left|\frac{1}{\Gamma(s(t))}\int_{\varsigma}^{t_1}[(t_1-\lambda)^{s(t)-1}-(t_2-\lambda)^{s(t)-1}]f\big(\lambda,y(\lambda),y(\lambda-\tau)\big)\,d\lambda\right|$$
$$+\left|\frac{1}{\Gamma(s(t))}\int_{t_1}^{t_2}(t_2-\lambda)^{s(t)-1}\,f\big(\lambda,y(\lambda),y(\lambda-\tau)\big)d\lambda\right|$$
$$\leq \frac{\|\mathfrak{I}y\|_{[0,\varsigma]}}{\Gamma(s(t))}\int_0^{\varsigma}[(t_1-\lambda)^{s(t)-1}-(t_2-\lambda)^{s(t)-1}]\lambda^{-\eta}\,d\lambda+\frac{K}{\Gamma(s(t))}\int_{\varsigma}^{t_1}[(t_1-\lambda)^{s(t)-1}-(t_2-\lambda)^{s(t)-1}]\,d\lambda$$
$$+\frac{K}{\Gamma(s(t))}\int_{t_1}^{t_2}(t_2-\lambda)^{s(t)-1}\,d\lambda$$
$$\leq |G(t_1)-G(t_2)|\frac{\|\mathfrak{I}y\|_{[0,\varsigma]}}{\Gamma(s(t))}+\frac{K}{\Gamma(s(t))}[2(t_2-t_1)^{s(t)}+(t_1-\varsigma)^{s(t)}-(t_2-\varsigma)^{s(t)}]$$

Since G(t) is continuous and by Cauchy's Convergence criterion, it follows that
$\lim_{t\to\zeta^-} y(t)=y^*$ exists.

Step 2: Here we will prove that $y(t)$ is continuable.
Since X is a closed subset, $(\varsigma, y^*)\in X$.
We have $y(\varsigma)=y^*$ and $y(t)\in C[0,\varsigma]$, we define the operator Θ as follows
$(\theta z)(t)=y_1+\frac{1}{\Gamma(s(t))}\int_{\varsigma}^{t}(t-\lambda)^{s(t)-1}f(\lambda,z(\lambda),z(\lambda-\tau))d\lambda$, where

$$y_1=y_0+\frac{1}{\Gamma(s(t))}\int_0^{\varsigma}(t-\lambda)^{s(t)-1}f(\lambda,z(\lambda),z(\lambda-\tau))d\lambda$$

where $z\in C[\varsigma,\varsigma+1]$ and $t\in[\varsigma,\varsigma+1]$
Let $W_a=\{(t,z): t\in[\varsigma,\varsigma+1], |x|\leq \max_{t\in[\varsigma,\varsigma+1]}|y_1(t)|+a\}$
Since f is continuous on W_a, we have $M=\max_{(t,z)\in W_a}|f(t,y(t),y(t-\tau)|$.
Let $W_h=\{z\in C[\varsigma,\varsigma+1]: \max_{t\in[\varsigma,\varsigma+1]}|z(t)-y_1(t)|\leq a, z(\varsigma)=y_1(\varsigma)\}$
Where $h^*=\min\left\{1,\left(\frac{M}{\Gamma(s(t)+1)a}\right)^{s(t)}\right\}$.
Thus Θ is completely continuous on W_a. Set $\{z_n\}\subseteq C[\varsigma,\varsigma+1], \|z_n-z\|_{[\varsigma,\varsigma+1]}\to 0$ as $n\to\infty$. Then we have

$$\|(\theta z_n)(t)-(\theta z)(t)\|=\left|\frac{1}{\Gamma(s(t))}\int_0^{\varsigma}(t-\lambda)^{s(t)-1}\left[f\big(\lambda,z_n(\lambda),z_n(\lambda-\tau)\big)-f\big(\lambda,z(\lambda),z(\lambda-\tau)\big)\right]d\lambda\right|$$
$$\leq \frac{h^{s(t)}}{\Gamma(s(t)+1)}\left\|f\big(\lambda,z_n(\lambda),z_n(\lambda-\tau)\big)-f\big(\lambda,z(\lambda),z(\lambda-\tau)\big)\right\|_{[\varsigma,\varsigma+h^*]}$$

Since f is continuous $\left\|f\big(\lambda,z_n(\lambda),z_n(\lambda-\tau)\big)-f\big(\lambda,z(\lambda),z(\lambda-\tau)\big)\right\|_{[\varsigma,\varsigma+h^*]}\to 0$ as $n\to\infty$.
Hence, $\|(\theta z_n)(t)-(\theta z)(t)\|_{[\varsigma,\varsigma+h^*]}\to 0$ as $n\to\infty$ shows that Θ is continuous.

Now, we will prove θW_h is continuous. For any $z\in W_h$, we have $(\theta z)(\varsigma)=y_1(\varsigma)$ and

$$|(\theta z)(t)-y_1|=\left|y_1+\frac{1}{\Gamma(s(t))}\int_{\varsigma}^{t}(t-\lambda)^{s(t)-1}f(\lambda,z(\lambda),z(\lambda-\tau))d\lambda-y_1\right|$$
$$=\left|\frac{1}{\Gamma(s(t))}\int_{\varsigma}^{t}(t-\lambda)^{s(t)-1}f(\lambda,z(\lambda),z(\lambda-\tau))d\lambda\right|$$
$$\leq \frac{M(t-\varsigma)^{s(t)}}{\Gamma(s(t)+1)}\leq \frac{Mh^{*s(t)}}{\Gamma(s(t)+1)}\leq a.$$

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Thus, $\Theta W_{h^*} \subset W_{h^*}$. If we set $\ell(t) = \frac{1}{\Gamma s(t)}\int_0^{\varsigma}(t-\lambda)^{s(t)-1}f(\lambda, z(\lambda), z(\lambda-\tau))d\lambda$. We have $\ell(t)$ is continuous on $[\varsigma, \varsigma+1]$. For all $z \in W_{h^*}, t_1, t_2 \in [\varsigma, \varsigma+h^*]$, we have

$$|(\Theta z)(t_1) - (\Theta z)(t_2)|$$
$$\leq \left|\frac{1}{\Gamma(s(t))}\int_0^{\varsigma}[(t_1-\lambda)^{s(t)-1} - (t_2-\lambda)^{s(t)-1}]\, f\big(\lambda, z(\lambda), z(\lambda-\tau)\big)d\lambda\right| + \left|\frac{1}{\Gamma(s(t))}\int_{\varsigma}^{t_1}[(t_1-\lambda)^{s(t)-1} - (t_2-\lambda)^{s(t)-1}] f\big(\lambda, z(\lambda), z(\lambda-\tau)\big)\, d\lambda\right| + \left|\frac{1}{\Gamma(s(t))}\int_{t_1}^{t_2}(t_2-\lambda)^{s(t)-1}\, f\big(\lambda, z(\lambda), z(\lambda-\tau)\big)d\lambda\right|$$
$$\leq |\ell_1(t_1) - \ell_2(t_2)| + \frac{M}{\Gamma(s(t)+1)}[2(t_2-t_1)^{s(t)} + (t_1-\varsigma)^{s(t)} - (t_2-\varsigma)^{s(t)}]$$

Since $\ell(t)$ is uniformly continuous on $[\varsigma, \varsigma+h^*]$ and from (3.5) we conclude that $\{(\Theta z)(t): z \in W_{h^*}\}$is equicontinuous. Thus Θ is continuous completely. Hence by Lemma 3, the operator Θ has a fixed point $\bar{y}(t) \in W_{h^*}$. that is

$$\bar{y}(t) = y_1 + \frac{1}{\Gamma(s(t))}\int_{\varsigma}^{t}(t-\lambda)^{s(t)-1}f(\lambda, \bar{y}(\lambda), \bar{y}(\lambda-\tau))d\lambda\,, \quad t \in [\varsigma, \varsigma+h] \tag{3.6}$$
$$= y_0 + \frac{1}{\Gamma(s(t))}\int_0^{\varsigma}(t-\lambda)^{s(t)-1}f(\lambda, \bar{y}(\lambda), \bar{y}(\lambda-\tau))d\lambda + \frac{1}{\Gamma(s(t))}\int_{\varsigma}^{t}(t-\lambda)^{s(t)-1}f(\lambda, \bar{y}(\lambda), \bar{y}(\lambda-\tau))d\lambda$$
$$= y_0 + \frac{1}{\Gamma s(t)}\int_0^{t}(t-\lambda)^{s(t)-1}f(\lambda, \bar{y}(\lambda), \bar{y}(\lambda-\tau))d\lambda$$

Where $\bar{y}(t) = \begin{cases} y(t)\ when\ t \in (0, \varsigma] \\ \bar{y}(t)\ when\ t \in [\varsigma, \varsigma+h^*] \end{cases}$

It follows that $\bar{y}(t) \in C[0, \varsigma+h^*]$ and

$$\bar{y}(t) = y_0 + \frac{1}{\Gamma(s(t))}\int_0^{t}(t-\lambda)^{s(t)-1}f(\lambda, \bar{y}(\lambda), \bar{y}(\lambda-\tau))d\lambda. \tag{3.7}$$

By Lemma 1, $\bar{y}(t)$ of (3.7), is a solution of (1.1) on $(0, \varsigma+h^*]$.
This is a contradiction to the fact that $y(t)$ is non-continuable.
Hence we proved the required result.

4. Global Solution

Theorem 4:
Assume that Hypothesis 1 holds. Consider $y(t)$ is a solution for (1.1) on $(0, \varsigma)$. For $\varepsilon > 0$, and if $y(t)$ is bounded on $[\varepsilon, \varsigma)$, then $\varsigma = +\infty$.

Lemma 5: [9, 10]
Let r be a real valued function defined on $[0, \beta] \times [0, \infty)$.
Assume that there exists $c > 0\ and\ s(t) \in (0,1)$, such that $r(t) \leq q(t) + c\int_0^t \frac{r(\lambda)}{(t-\lambda)^{s(t)}}d\lambda$,
where $q(t) > 0$ is a locally integrable function in $[0, \beta]$. Then there exists $h = h\big(s(t)\big)$, such that for $t \in [0, \beta]$, we have $r(t) \leq q(t) + hc\int_0^t \frac{q(\lambda)}{(t-\lambda)^{s(t)}}d\lambda$.

Theorem 5:
Assume that (H1) holds and consider three continuous functions $h(t), i(t)\ and\ j(t)$ defined on $[0, \infty)X[0, \infty)$, such that $|f(t, y(t), y(t-\tau)| \leq h(t)i(|y|) + j(t), where\ i(s) \leq s\ for\ some\ s \in [0, \infty)$. Then there exists a solution of (1.1) in $C[0, \infty)$.

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Proof:

We can easily conclude the local existence of the solution of (1.1) by using Theorem1.

By Lemma 1, $y(t)$ satisfies the following equation

$y(t) = y_0 + \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} f(\lambda, y(\lambda), y(\lambda-\tau))d\lambda.$

Let the maximum interval of $y(t)$ as $[0,\varsigma), where\ \varsigma < \infty$.

Then

$$|y(t)| = \left| y_0 + \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} f(\lambda, y(\lambda), y(\lambda-\tau))d\lambda \right|$$

$$\le y_0 + \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1}(h(\lambda)\, i(|y|) + j(\lambda))d\lambda$$

$$\le y_0 + \frac{\|h\|_{[0,\varsigma)}}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1}(h\,|y|)d\lambda + \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} j(\lambda)d\lambda.$$

By taking

$r(t) = |y(t)|,\ q(t) = y_0 + \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} j(\lambda)d\lambda\ \ and\ c = \frac{\|h\|_{[0,\varsigma)}}{\Gamma(s(t))}$

by Lemma 5, $r(t) = |y(t)|$ is bounded on $[0,\varsigma)$. Thus, for any $\varepsilon \in (0,\varsigma), y(t)$ is bounded on $[\varepsilon,\varsigma)$. By theorem 4, the solution of (1.1) exists on $(0,\infty)$.

The following theorem ensures the existence and uniqueness of the global solution of (1.1) on $\mathbb{R}^+$.

Theorem 6:

Assume that (H1) holds, and a continuous function $p(t) > 0$ exists and defined on $[0,\infty)$, such that $|f(t,y) - f(t,\bar{y})| \le p(t)|y-\bar{y}|$, then unique solution of (1.1) exists in $C[0,\infty)$.

5. Ulam Stability Results

Let us consider the Ulam stability for (1.1) as follows. Let $\varepsilon > 0$ and φ be a continuous function defined on $[0,\infty) \to \mathbb{R}^+$. Now, Consider the following inequalities:

$$\left| {}_C D_{0,t}^{s(t)} y(t) - f(t, y(t), y(t-\tau)) \right| < \varepsilon \tag{5.1}$$

$$\left| {}_C D_{0,t}^{s(t)} y(t) - f(t, y(t), y(t-\tau)) \right| < \varphi(t) \tag{5.2}$$

$$\left| {}_C D_{0,t}^{s(t)} y(t) - f(t, y(t), y(t-\tau)) \right| < \varepsilon\varphi(t) \tag{5.3}$$

Definition 5:

The Initial value Problem is Ulam-Hyers Stable, if there its a real number $c_j > 0$, such that for each $\varepsilon > 0$ and for each solution $y \in C[0,\infty)$ of (5.1), there exists a solution $z \in C[0,\infty)$ of (1.1) with $|y(t) - z(t)| \le \varepsilon c_j$

Definition 6:

The Initial value Problem is Ulam-Hyers Stable, if there is a real number $c_j \in (\mathbb{R}^+, \mathbb{R}^+)$ with $c_j(0) = 0$, for each $\varepsilon > 0$ and for each solution of $y \in C$ of (5.2), then there exists a solution $z \in C$ of (1.1) with $|y(t) - z(t)| \le \varepsilon c_j$.

Definition 7:

If there is a number $c_{j,\varphi} \in \mathbb{R}^+$, for each $\varepsilon > 0$ and for each solution of $y \in C$ of (5.3), then there exists a solution $z \in C$ of (1.1) with $|y(t) - z(t)| \le \varepsilon c_{j,\varphi}\varphi(t)$, then IVP (1.1) is Ulam–Hyers–Rassias stable, with respect to φ.

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Definition 8:
If there is a number $c_{j,\varphi} \in \mathbb{R}^+$, for each $\varepsilon > 0$ and for each solution of $y \in C$ of (5.2) then there exists a solution $z \in C$ of (1.1) with $|y(t) - z(t)| \leq c_{j,\varphi}\varphi(t)$, then IVP (1.1) is the generalized Ulam–Hyers–Rassias stable, with respect to φ.

Hypothesis 3: [H3]
If we assume $\varphi(t)$ is an increasing function and $\varphi \in C[0,\infty)$ then there exists $\chi_\varphi > 0$, such that
$\frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1}\varphi(\lambda)d\lambda \leq \chi_\varphi \varphi(t), \quad t \in [0,\infty)$

Lemma 6: [11]
Let $x(t) and\ y(t)$ be a continuous function defined on $[0,T] \times [0,\infty)$ where $T \leq \infty$.
If y is increasing and there are constants $\mu \geq 0\ and\ p \geq 0$, such that

$$x(t) \leq y(t) + \mu \int_0^t (t-\lambda)^{p-1} x(\lambda) d\lambda, \qquad t \in [0,T),$$

Then

$$x(t) \leq y(t) + \int_0^t [\sum_{k=0}^{\infty} \frac{(\mu\Gamma(p))^k}{\Gamma(kp)} (t-\lambda)^{p-1} y(\lambda) d\lambda, \qquad t \in [0,T).$$

If $y(t) = c$ is a constant on $t \in [0,T)$, then we have
$x(t) \leq cE_p(\mu\Gamma(p)t^p), t \in [0,T),$
where $E_p(\cdot)$ is the Mittag–Leffler function.
Theorem 7:
If (H3) is satisfied, then IVP (1.1) is the generalized Ulam–Hyers–Rassias stable.
Proof:
Suppose that u is a solution of (5.2) on $C[0,\infty)$, and we assume that z is a solution of (1.1). Thus, we have

$$\left| y(t) - y_0(t) - \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} f(\lambda, y(\lambda), y(\lambda-\tau)) d\lambda \right|$$

$$\leq \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1}\varphi(\lambda)d\lambda \ \leq \chi_\varphi\varphi(t)$$

From the above relations, it follows that

$$|y(t) - z(t)| \leq \left| y(t) - y_0(t) - \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} f(\lambda, y(\lambda), y(\lambda-\tau)) d\lambda \right|$$
$$+ \frac{1}{\Gamma(s(t))}\int_0^t (t-\lambda)^{s(t)-1} \left| (f(\lambda, y(\lambda), y(\lambda-\tau)) - (f(\lambda, z(\lambda), z(\lambda-\tau)) \right| d\lambda$$

$\leq \chi_\varphi\varphi(t) + \int_0^t (t-\lambda)^{s(t)-1} |y(\lambda) - z(\lambda)|\, d\lambda.$
By Lemma 6, there exists a constant L^* independent of $\chi_\varphi\varphi(t)$, such that
$|y(t) - z(t)| \leq L^*\chi_\varphi\varphi(t)$. Hence proved.

6. Example

We verify our result by means of the following examples
Example 1:
Consider the following IVP of VOFDDE

$$\begin{cases} {}_C D_{0,t}^{s(t)} y(t) = \frac{|y(t-\tau)|}{7(1+|y(t)|^4)}, 0 < s(t) < 1 \\ \quad y_0 = 0, y \in \mathbb{R}, t \in (0,2) \end{cases} \tag{6.1}$$

$with\ s(t) = \frac{1}{14}(8 + 3t).$

In this example the function $f\big(t, y(t), y(t-\tau)\big) = \frac{|y(t-\tau)|}{7(1+|y(t)|^4)}$ is continuous with respect to $y \in \mathbb{R}$.

Also, $|\ f(t, y_1) - f(t, y_2)\ | \leq L\ |\ y_1 - y_2\ |\ with\ L = \frac{1}{7} > 0.$

Here $f\big(t, y(t), y(t-\tau)\big)$ satisfies (H1) and (H2). Hence the IVP (6.1) has a unique solution.

Example 2:

Consider the following IVP of VOFDDE

$$\begin{cases} {}_C D_{0,t}^{s(t)} y(t) = |t|e^{ty} + t sin(t-2), 0 < s(t) < 1 \\ \qquad y_0 = 0, y \in \mathbb{R}, t \in (0,1) \end{cases} \tag{6.2}$$

$with\ s(t) = 0.51 + 0.49t.$

This also satisfies (H1) and (H2). Hence the IVP (6.2) has a unique solution.

7. Conclusion

In this paper we give the local existence and uniqueness theorems for the equation (1.1). Also, we proved the continuation theorem to establish the global existence of VOFDDE.

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