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Intuitionistic fuzzy cone-b metric space and some fixed point results

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KEYWORDS

b-metric space, IFCbMS, fixed point

ABSTRACT: In this present work a new space named Intuitionistic Fuzzy Cone-b Metric Space (in short, IFCbMS) is introduced. In order to strengthen the concept, Banach contraction principle and other results are also presented in this new setting of IFCbMS. Moreover, our results are the enhancement and improvement on the previous research findings.

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AMS Subject Classification: 54H25, 47H10.

1. Introduction

Advancement in day to day life leads to the generation of uncertainty. Mathematical sciences give answers to these uncertainties in terms of fuzzy sets (FS). The

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origination of FS was given by L. Zadeh [1] in 1965. It paved the path of new beginnings for many authors which leads to the generation of further branches of FS like Intuitionistic FS (IFS) [16], Neutrosophic FS, Picture FS, Hesitant FS etc. All these spaces differ a little bit from each other depending upon the situations given. Firstly the conception of b-metric space was brought by Czerwik [17] which was further redefined by Bakhtin in 1989. Then in 2007 [22] came the Idea of cone metric space (CMS) into light that gave different authors a lot more into the branch of fuzziness. Afterwards Hussain and Shah [20] in 2011 developed the concept of cone b-metric space (CbMS).

After that Oner, Kandemire and Tanay [22] in 2011 used the idea of FMS ([2], [3]) and CMS leading to the development of FCMS. Vishal Gupta and Surjeet Singh Chauhan [20] in 2020 gave the concept of FCbMS. Now in this work we have introduced the concept of IFCbMS.

2. Preliminaries

Definition 2.1[2] An operator$\circ$: $U \times U \rightarrow U$ is known as continuous t-norm when:-

1. $\circ$ being continuous and associative;
2. $\circ$ being commutative;
3. $g \circ 1 = g$;
4. $g \circ h \leq v \circ w$, when it is given that $g \leq v$ and $h \leq w$, where $g, h, v, w \in U$, where $U = [0, 1]$.

Definition 2.2[16] An operator$\triangle$: $U \times U \rightarrow U$ is taken as a continuous t-co-norm when:-

1. $\triangle$ being commutative and associative;
2. $\triangle$ being continuous;
3. $e \triangle 0 = e$;
4. $e \triangle d \leq u \triangle s$, when it is given that $e \leq u$ and $d \leq s$, where $e, d, u, s \in U$, where $U = [0, 1]$.

Definition 2.3[13] A three-tuple $(U, L, \circ)$, where$\circ$ is t-normand L is FS defined on $U^2 \times [0, \infty)$ is called a FMS when:-

1. $L(g, b, 0) > 0$;
2. $L(g, b, f) = 0$ iff$g = b$, where $f > 0$;
3. $L(g, b, f) = L(b, g, f)$;
4. $L(g, b, f) \circ L(b, r, s) \leq L(g, r, f + s)$;
5. $L(g, b, \cdot) : [0, \infty) \rightarrow [0, 1]$ being left continuous;
6. $L(g, b, f) = 1$, where $g, b, r \in U$ and $s, f > 0$.

Definition 2.4([3]) A five-tuple set $(U, L, Q, \circ, \triangle)$, where U is any random universal set, L and Q are FS on $U^2 \times [0, \infty)$,$\circ$ being a t-norm, and $\triangle$ being a t-co-norm is called an Intuitionistic fuzzy metric space (IFMS) when:-

1. $L(u, d, f) + Q(u, d, f) \leq 1$, where $u, d \in U$ and $f > 0$;
2. $L(u, d, 0) > 0$, where $u, d \in U$;
3. $L(u, d, f) = 1$, where $u, d \in U$ and $f > 0$ iff $u = d$;
4. $L(u, d, f) = L(d, u, f)$, where $u, d \in U$ and $f > 0$;
5. $L(u, d, f) \circ L(d, r, s) \leq L(u, r, f + s)$, where $u, d, r \in U$ and $s, f > 0$;
6. for $e, d \in U$, $L(u, d, \cdot) : [0, \infty) \rightarrow [0, 1]$ being left continuous;
7. $\lim_{f \to \infty} L(u, d, f) = 1$, where $u, d \in U$ and $f > 0$;
8. $Q(u, d, 0) < 1$, where $u, d \in U$;
9. $Q(u, d, f) = 0$ iff $u = d$ where $u, d \in U$ and $f > 0$;
10. $Q(u, d, f) = Q(d, u, f)$, where $u, d \in U$ and $f > 0$;
11. $Q(u, d, f) \triangle Q(d, r, s) \geq Q(u, r, f + s)$, where $u, d, r \in U$ and $s, f > 0$;

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12. For u, d ∈U, Q (u, d, ·) : [0, ∞) → [0, 1] being right continuous;
13. $\lim_{f\to\infty}$ Q (u, d, f) = 0, where u, d∈ U.

Definition 2.5[20] Let S be a subset of B. Then S is called a cone when:-

1. S is closed, non-null and S ≠ {0};
2. if c, d ∈ [0, ∞) and u, v ∈ S then(cu + dv) ∈ S;
3. if both u and –u ∈S then u = 0.

Here, int(S) denotes the collection of all those elements of S which lies in the interior of S.

Definition 2.6 [22] A three tuple (U, D, ∘), whereS is a cone of B, U is a randomly chosen set, ∘ is a t-norm and D is a FS on U^2×int (S) is called FCMS when for w, f, h ∈ U and t, s ∈ int(S):-

1. D(w, f, t) > 0 and D(w, f, t) =1 iff w = f;
2. D(w, f, t) = D(f, w, t);
3. D(w, h, t+ s) ≥ D(w, f, t) ∘ D(f, h, s);
4. D(w, f, ·): int(S)⟶ [0,1] is continuous.

Definition 2.7([6], [7]) Let U be an arbitrary set,∘ be a t-norm and k ≥1 be a real number. A FS H on U×U×(0, ∞)is taken as a b-FMS if for anyu, g, f ∈ U and t, s > 0:-

1. H(u, g, 0) > 0;
2. H(u, g, t) = 0 if and only if u = g;
3. H(u, g, t) = H(g, u, t);
4. $H(u, f, t + s) \geq H\left(u, g, \frac{t}{k}\right) \circ H(g, f, \frac{s}{k});$
5. H(u, g, ·):(0, ∞)⟶(0,1] is continuous.

Definition 2.8[20] A three tuple (U, H, ∘) is a FCbMS if S is a cone of B, U is any arbitrarily taken set,∘ is a t-norm and H being a FS on U^2× int (S) and for u, d, f ∈ U and t, s ∈ int (S), λ≥1:-

1. H(u, d, t) > 0, H(u, d, 0) = 0 iff u = d;
2. H(u, d, t) =1 for u, d > 0;
3. H(u, d, t) = H(d, u, t);
4. H(u, f, λ(t+ s)) ≥ H(u, d, t) ∘H(d, f, s);
5. H(u, d,·): int (S)⟶ [0,1] is continuous and $\lim_{t\to\infty}$H(u, d, t)=1.

Definition 2.9 [21] Let B= R^2 be a real Banach space (BS). Then S= {(r_1, r_2): r_1, r_2≥ 0} a subset of B with normal constant k = 1. Let Ube a random set,∘ is a t-normdefined by u∘d = u d and H: U^2 ×int (S) ⟶ [0, 1]be a fuzzy set defined by $H(u, d, t) = \frac{1}{e^{\frac{\|u-d\|}{\|\lambda t\|}}}$for all u,d ∈ U and t ≥ 0 then (U, L, ∘) represents a FCbMS.

Definition 2.10[21]A 5-tuple (U, H, E, ∘, △), where U is any random set, H and E are FS on U^2× int (S) ⟶ [0, 1], ∘ is a t-norm and △ is a t-co-normis called an Intuitionistic fuzzy cone metric space (IFCMS)when:-

1. H(u, d, t) + E(u, d, t) ≤ 1;
2. H(u, d,t) > 0, H(u, d, 0) = 0;
3. H(u, d, t) = 1 iff u=d;
4. H(u, d, t) = H(d, u, t);
5. H(u, d, t) ∘H(d, f, s) ≤ H(u, f, t + s);
6. H(u, d, ·): int(S)⟶ [0, 1] is continuous;
7. E(u, d, t) > 0;
8. E(u, d, t) = 0 iff u=d;
9. E(u, d, t) = E(d, u, t);
10. E(u, d, t) △ E(d, f, s) ≥ E(u, f, t + s);
11. E(u, d, ·): int(S)⟶ [0, 1] is continuous.

Definition 2.11:- IFCbMS (Intuitionistic fuzzy cone-b metric space)

Let U be a non-null random Set, ∘be a t-norm, △ be a t-co-norm, H and E are FS defined on U^2 ×int(S), where S is a cone of B (a real BS). A 6-tuple

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(U, H, E, ∘, △, λ) is taken as an IFCbMS when for all u, d, f ∈ U and t, s ∈ int (S), λ ≥ 1

1. H(u, d, t) + E(u, d, t) ≤ 1;
2. H(u, d, t)> 0, H(u, d, 0) = 0;
3. H(u, d, t) =1 iff u = d;
4. H(u, d, t) = H(d, u, t);
5. H(u, d, t)∘ H(d, f, s) ≤ H(u, f, λ (t + s)), t, s ≥ 0 ;
6. H(u, d, ·): int(S)→[0, 1] is continuous and $\lim_{t\to\infty} H(u, d, t) = 1$;
7. E(u, d, t)> 0, E(u, d, 0) = 0;
8. E(u, d, t) = 0 iff u = d ;
9. E(u, d, t) = E(d, u, t);
10. E(u, d, t)△ E(d, f, s) ≥ E(u, f, λ(t + s));
11. E(u, d, ·): int(S)→[0, 1] is continuous and $\lim_{t\to\infty} E(u, d, t) = 0$.

Example 2.12:- Let B= R^2 and S = {(s, s); s, s, ≥ 0} be a subset of Band S be a cone possessing normal constant k=1. Let U =R, u∘ d = min{u, d}, u△ d = max {u, d}, H, E :U^2 × int (S)→[0,1]given by

H(u, d, t)=$\frac{1}{e^{\frac{|u-d|}{||\lambda t||}}}$andE(u,d,t)=$\frac{|u-d|}{e^{\frac{|u-d|}{||\lambda t||}}}$

1. H(u, d, t) + E(u, d, t)=$\frac{1}{e^{\frac{|u-d|}{||\lambda t||}}}+\frac{|u-d|}{e^{\frac{|u-d|}{||\lambda t||}}}=\frac{1+|u-d|}{e^{\frac{|u-d|}{||\lambda t||}}}$.

2. H(u, d, t) = $\frac{1}{e^{\frac{|u-d|}{||\lambda t||}}}$>0,

   H(u, d, 0) = $\frac{1}{e^{\frac{|u-d|}{|0|}}}=\frac{1}{e^{\infty}}=\frac{1}{\infty}$= 0.

3. H(d, d, t) = $\frac{1}{e^{\frac{|d-d|}{||\lambda t||}}}=\frac{1}{e^{0}}$=1 atu=d.

4. H(u, d, t)=$\frac{1}{e^{\frac{|u-d|}{||\lambda t||}}}$=

H(d, u, t)=$\frac{1}{e^{\frac{|d-u|}{||\lambda t||}}}=\frac{1}{e^{\frac{|u-d|}{||\lambda t||}}}$= H(u, d, t).

5. s ≤ t + s ≤ λ(t + s) and t ≤ t + s≤ λ(t+s) and ||s|| ≤ ||λ(t + s)|| and ||t|| ≤||λ(t + s)||

   this implies $\frac{||\lambda(t+s)||}{||s||}\geq 1$ and$\frac{||\lambda(t+s)||}{||t||}\geq 1$,

   now, |u – f| ≤ |u – d| +|d – f| so,

$|u-f| \leq |u-d|\frac{||\lambda(t+s)||}{||t||}+|d-f|\frac{||\lambda(t+s)||}{||s||}$,

   we have $\frac{|u-f|}{||\lambda(t+s)||}\leq\frac{|u-d|}{||t||}+\frac{|d-f|}{||s||}$,

   or $\frac{1}{e^{\frac{|u-f|}{||\lambda(t+s)}}}\geq\frac{1}{e^{\frac{|u-d|}{||t||}}}+\frac{1}{e^{\frac{|d-f|}{||s||}}}$therefore

   H(u, f, λ(t+ s)) ≥ H(u, d, t)∘H(d, f, s).

6. H (u, d, t) = $\frac{1}{e^{\frac{|u-d|}{||\lambda t||}}}$ being an exponential function is continuous.

$$\lim_{t\to\infty} H(u, d, t) = \lim_{t\to\infty}\frac{1}{e^{\frac{|u-d|}{||\lambda t||}}} = e^{-\frac{|u-d|}{|\lambda|||t||}}$$

$$=1-\frac{|u-d|}{|\lambda|||t||}+\frac{|u-d|^2}{|\lambda|||t||}-\frac{|u-d|^3}{|\lambda|||t||}+ \ldots\ldots$$

which is GP series with common ratio $\frac{|u-d|}{|\lambda|||t||}$ which is convergent when $\frac{1}{|\lambda|}$<1 as t→∞

therefore $\lim_{t\to\infty} e^{-\frac{|u-d|}{|\lambda|||t||}}$ =1.

7. E(u,d,t)=$\frac{|u-d|}{e^{\frac{|u-d|}{||\lambda t||}}}$>0,

   E(u, d, 0)= $\frac{|u-d|}{e^{\frac{|u-d|}{||0||}}}$> 0,

   $=\frac{|u-d|}{e^{\infty}}=\frac{|u-d|}{\infty}$= 0.

8. E(u, d, t)= $\frac{|u-d|}{e^{\frac{|u-d|}{||\lambda t||}}}$>0and at u=d,E(d, d, t)= $\frac{|d-d|}{e^{0}}$= 0.

9. E(u, d, t)= $\frac{|u-d|}{e^{\frac{|u-d|}{||\lambda t||}}}=\frac{|d-u|}{e^{\frac{|d-u|}{||\lambda t||}}}$=E(d, u, t).

10. To prove:- E (u, d, t) △ E(d, f, s) ≥ E(u, f, λ(t + s))

$s \le t + s \le \lambda(t + s), t \le (t + s) \le \lambda(t + s)$,

$\|s\| \le \| \lambda(t + s)\|$, $\|t\| \le \|\lambda(t + s)\|$,

$\frac{\|\lambda(t+s)\|}{\|s\|} \ge 1$, $\frac{\|\lambda(t+s)\|}{\|t\|} \ge 1$,

Now, $|u - f| \le |u - d| + |d - f|$,

(2.1)

$|u - f| \le |u - d| \frac{\|\lambda(t+s)\|}{\|t\|} + |d - f| \frac{\|\lambda(t+s)\|}{\|s\|}$, we get

$\frac{|u-f|}{\|\lambda(t+s)\|} \le \frac{|u-d|}{\|t\|} + \frac{|d-f|}{\|s\|}$,

therefore, $e^{\frac{|u-f|}{\|\lambda(t+s)\|}} \le e^{\frac{|u-d|}{\|t\|}} + e^{\frac{|d-f|}{\|s\|}}$

(2.2)

Using (2.1) and (2.2) $\frac{|u-f|}{e^{\frac{|u-f|}{\|\lambda(t+s)\|}}} \le \frac{|u-d|+|d-f|}{e^{\frac{|u-d|}{\|t\|}} + e^{\frac{|d-f|}{\|s\|}}}$,

This gives, $\frac{|u-f|}{e^{\frac{|u-f|}{\|\lambda(t+s)\|}}} \le \frac{|u-d|}{e^{\frac{|u-d|}{\|t\|}} + e^{\frac{|d-f|}{\|s\|}}} + \frac{|d-f|}{e^{\frac{|u-d|}{\|t\|}} + e^{\frac{|d-f|}{\|s\|}}} \le \frac{|u-d|}{e^{\frac{|u-d|}{\|t\|}}} + \frac{|d-f|}{e^{\frac{|d-f|}{\|t\|}}}$,

Thus $E(u, f, \lambda(t + s)) \le E(u, d, t) \triangle E(d, f, s)$.

11. $E(u, d, \cdot) = \frac{|u-d|}{e^{\frac{|u-d|}{\|\lambda t\|}}}$ is continuous as it is a composite fraction of polynomial and exponential function. $\lim_{t\to\infty} E(u, d, t) = 0$ as $\lim_{\theta\to\infty} \frac{\theta}{e^{\theta}} = 0$ [by L' Hospital rule].

3. Main Results

Theorem 3.1(IFCbMS and Banach contraction theorem)

Let $(U, H, E, \circ, \triangle, \lambda)$ be an IFCbMS. Let $T: U \to U$ be a mapping which holds the contractive conditions

$$H(Tp, Tm, kt) \ge H(p, m, t), \quad (3.1)$$

$$E(Tp, Tm, kt) \le E(p, m, t). \quad (3.2)$$

for all $p, m \in U$, where $0 < k < 1$, $\circ$ be a t-norm, $\triangle$ be a t-co-norm, D and E are FS defined on $U^2 \times int(S)$, where S is a cone of B (a real BS). Then T exhibits a unique fixed point.

Proof: Let $p_o \in U$ be an arbitrarily chosen element and let $\{w_q\}$ be a sequence in U such that, $p_q = T^q p_o (q \in U)$. Then

$H(p_q, p_{q+1}, kt) = H(T^q p_o, T^{q+1} p_o, kt) \ge H(T^{q-1} p_o, T^q p_o, t)$

$= H(p_q, p_{q-1}, t) \ge H(T^{q-2} p_0, T^{q-1} p, t/k)$

$= H(p_{q-2}, p_{q-1}\ t/k) \dots \ge H(p, p, t/k^{q-1})$.

clearly, $1 \ge H(p, p_{q+1}, kt) \ge H(p_0, p_1, t/k^{q-1}) \to 1$, whenq $\to \infty$.

So, $\lim_{q\to\infty} H(p_q, p_{q+1}, kt) = 1$,

we have $E(p_q, p_{q+1}, kt) = E(T^q p_0, T^{q+1} p_0, kt) \le E(T^{q-1} p_0, T^q p_0, t) = E(p_{q-1}, p_{q-1}, t)$

$\le E(T^{q-2} p_0, T^{q-1} p_0, t/k) = E(p_{q-2}, p_{q-1}, t/k) \dots \le E(p_0, p_1, t/k^{q-1})$,

for all q and $t > 0$. Clearly, $0 \le E(p_q, p_{q+1}, kt) \le E(p_0, p_1, t/k^{q-1}) \to 0$, when $q \to \infty$.

So, $\lim_{q\to\infty} E(p_q, p_{q+1}, kt) = 0$. Let $\tau_q(t) = H(p_q, p_{q+1}, t)$ and $\mu_q(t) = E(p_q, p_{q+1}, t)$, $t > 0$.

Next, we claim that the sequence $\{p_q\}$ is a Cauchy sequence. If not, then for $0 < \epsilon < 1$, we can define two sequences $\{r(q)\}$ and $\{s(q)\}$ so that for every $q \in U \cup \{0\}$, $t > 0$, $r(q) > s(q) \ge q$,

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$H\big(p_{r(q)}, p_{s(q)}, t\big) \le 1-\epsilon$ and $E\big(p_{r(q)}, p_{s(q)}, t\big) \ge \epsilon$,

$H\big(p_{r(q)-1}, p_{s(q)-1}, t\big) > 1-\epsilon$, $H\big(p_{r(q)-1}, p_{s(q)}, t\big) > 1-\epsilon$.

$E\big(p_{r(q)-1}, p_{s(q)-1}, t\big) < \epsilon$, $E\big(p_{r(q)-1}, p_{s(q)}, t\big) < -\epsilon$.

Now, $1-\epsilon \ge H\big(p_{r(q)}, p_{s(q)}, t\big) \ge H\big(p_{r(q)-1}, p_{s(q)}, t/2\lambda\big) \circ H\big(p_{r(q)-1}, p_{s(q)}, t/2\lambda\big)$

$> \tau_{r(q)-1}(t/2\lambda) \circ (1-\epsilon)$,

$\epsilon \le E\big(p_{r(q)}, p_{s(q)}, t\big) \le E\big(p_{r(q)-1}, p_{s(q)}, t/2\lambda\big) \triangle E\big(p_{r(q)-1}, p_{s(q)}, t/2\lambda\big)$

$< \zeta_{r(q)-1}((t/2\lambda) \triangle \epsilon)$.

Since, $\tau_{r(q)-1}(t/2\lambda) \to 1$ as $q\to\infty$ and $\zeta_{s(q)-1}(t/2\lambda) \to 0$ as $q \to \infty$. Therefore for $q\to\infty$, we have

$1-\epsilon \ge H\big(p_{r(q)}, p_{s(q)}, t\big) > 1-\epsilon$

and $\epsilon \le E\big(p_{r(q)}, p_{s(q)}, t\big) < \epsilon$.

Clearly, this generates a contradiction. Hence $\{p_q\}$ is a Cauchy sequence in U. Since U is complete so there exist a point m in U such that $\lim_{q\to\infty} p_q = m$.

Now, $H(m, f, t) \ge H(m, p_{q+1}, t/2\lambda) \circ H(p_{q+1}, Tm, t/2\lambda)$

$= H(m, p_{q+1}, t/2\lambda) \circ H(Tp_q, Tm, t/2\lambda)$

$\ge H(m, p_{q+1}, t/2\lambda) \circ H(p_q, m, t/2\lambda k)$.

The case when $q\to\infty$ we have $H(m, Tm, t) \ge 1$

and $E(m, Tm, t) \le E(m, p_{q+1}, t/2\lambda) \triangle E(p_{q+1}, Tm, t/2\lambda)$

$= E(m, p_{q+1}, t/2\lambda) \triangle E(Tp_{q+1}, Tm, t/2\lambda)$

$\le E(m, p_{q+1}, t/2\lambda) \triangle E(p_{q+1}, m, t/2\lambda k)$.

on $q\to\infty$, we have $E(m, Tm, t) \le 0$.

By the definition of IFCbMS, we have m = Tm.

To claim uniqueness, we suppose m and f be two fixed points of the mapping T, then m = Tm, f = Tf and

$1 \ge H(d, f, t) = H(Td, Tf, t)$

$\ge H(d, f, t/k) = H(Td, Tf, t/k) \ge H(d, f, t/k^2) \ge \dots$

$\ge H(d, f, t/k^n) \to 1$ as $n\to\infty$.

also, $0 \le E(m, f, t) = E(Tm, Tf, t)$

$\le E(m, f, t/k) = E(Tm, Tf, t/k) \le E(m, f, t/k^2) \le \dots$

$\le E(m, f, t/k^n) \to 0$ as $n\to\infty$.

by the definition of IFCbMS, we have m = f.

Theorem 3.2:- Let $(U, H, E, \circ, \triangle, \mu)$ be a complete IFCbMS with $\circ$ is a t-norm defined by $b \circ h = \min\{b, h\}$ and $\triangle$ is a t-co-norm given by $b \triangle h = \max\{b, h\}$, H and E are FS defined on $U^2 \times \mathrm{int}(S)$, where S is a cone of B (a real BS). Also suppose that $H(b, h, \cdot)$ is strictly increasing and $E(b, h, \cdot)$ is strictly decreasing respectively. Let $A: U\to U$ be a self map which satisfies the following conditions for all $d, b \in U$.

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$H(Ab, Ah, kt) \geq H(b, Ab, t) \circ H(h, Ah, t)$ (3.1)

$E(Ab, Af, kt) \leq E(b, Ab, t) \triangle E(f, Af, t)$, where $t > 0, 0 < k < 1$. (3.2)

Then A acquires a unique fixed point.

Proof: Let $b_0 \in U$ be an arbitrary point. Consider a sequence $\{b_q\} = \{Ab_{m-1}\}$ of points in U.

Then $H(b_m, b_{m+1}, kt) = H(Ab_{m-1}, Ab_m, kt) \geq H(b_{m-1}, Ab_{m-1}, t) \circ H(b_m, Ab_m, t)$

$= H(b_{m-1}, b_m, t) \circ H(b_m, b_{m+1}, t)$,

Since $H(b, h, \cdot)$ is strictly increasing function, $kt < t$ and if

$\min\{H(b_{m-1}, b_m, t), H(b_m, b_{m+1}, t)\} = H(b_m, b_{m+1}, t)$,

this is a contradiction.

$H(b_m, b_{m+1}, kt) \geq H(b_m, b_{m+1}, t)$.

Therefore, $H(b_m, b_{m+1}, kt) \geq H(b_{m-1}, b_m, t) = H(Ab_{m-2}, Ab_{m-1}, t)$

$\geq H(b_{m-1}, Ab_{m-1}, t/k) \circ H(b_{m-2}, Ab_{m-2}, t/k)$

$= H(b_{m-1}, b_m, t/k) \circ H(b_{m-2}, b_{m-1}, t/k)$

$= H(b_{m-2}, b_{m-1}, t/k) \cdots$

$\geq H(b_0, b_1, t/k^{m-1})$.

Obviously, $1 \geq H(b_m, b_{m+1}, kt) \geq H(b_0, b_1, t/k^{m-1}) \rightarrow 1$, when $m \rightarrow \infty$.

Thus $\lim_{m\rightarrow\infty} H(b_m, b_{m+1}, kt) = 1$.

Now, $E(b_m, b_{m+1}, kt) = E(Ab_{m-1}, Ab_m, kt) \leq E(b_{m-1}, Ab_{m-1}, t) \triangle E(b_m, Ab_m, t)$

$= E(b_{m-1}, b_m, t) \triangle E(b_m, b_{m+1}, t)$.

Since, $E(b, h, \cdot)$ is strictly decreasing function, $kt < t$, by the same argument $E(b_m, b_{m+1}, kt) \leq E(b_m, b_{m+1}, t)$ is not possible.

Therefore $E(b_m, b_{m+1}, kt) \leq E(b_{m-1}, b_m, t) = E(Ab_{m-2}, Ab_{m-1}, t)$

$\leq E(b_{m-1}, Ab_{m-1}, t/k) \triangle E(b_{m-2}, Ab_{m-2}, t/k)$

$= E(b_{m-1}, b_m, t/k) \triangle E(b_{m-2}, b_{m-1}, t/k) = E(b_{m-2}, b_{m-1}, t/k) \cdots$

$\leq E(b_0, b_1, t/k^{m-1})$.

Clearly, $0 \leq E(b_m, b_{m+1}, kt) \leq E(b_0, b_1, t/k^{m-1}) \rightarrow 0$ when $m\rightarrow\infty$. Hence, $\lim_{m\rightarrow\infty} E(b_m, b_{m+1}, kt) = 0$.

Let $q(t) = H(b_m, b_{m+1}, t)$ and $\partial_q(t) = E(b_m, b_{m+1}, t)$ for all $m \in E \cup \{0\}$, $t > 0$.

Clearly, $\lim_{m\rightarrow\infty} \tau_m(t) = 1$, and $\lim_{m\rightarrow\infty} \partial_m(t) = 0$. Next, we show that the sequence $\{b_m\}$ is a Cauchy sequence. If not, then we can search two sequences $\{r(q)\}$ and $\{s(q)\}$ for $0 < \epsilon < 1$, so that for every $q \in U \cup \{0\}$, $t > 0$, $r(q) > s(q) \geq q$, $H(b_{r(q)}, b_{s(q)}, t) \leq 1-\epsilon$ and $E(b_{r(q)}, b_{s(q)}, t) \geq 1-\epsilon$

and $H(b_{r(q)-1}, b, t) > 1-\epsilon$, $H(b_{r(q)-1}, b_{s(q)}, t) > 1-\epsilon$

and $E(b_{r(q)-1}, b_{s(q)-1}, t) < 1-\epsilon$, $E(b_{r(q)-1}, b_{s(q)}, t) < 1-\epsilon$.

as $1-\epsilon \geq H(b_{r(q)}, b_{s(q)}, t)$

$\geq H(b_{r(q)-1}, b_{s(q)}, t/2\mu) \circ H(b_{r(q)-1}, b_{s(q)}, t/2\mu)$

$> \tau_{r(q)-1}(t/2\mu) \circ (1-\epsilon)$

and $\epsilon \leq E(b_{r(q)}, b_{s(q)}, t)$

$\leq E(b_{r(q)-1}, b_{s(q)}, t/2\mu) \triangle E(b_{r(q)-1}, b_{s(q)}, t/2\mu)$

$>\partial_{r(q)-1}(t/2\mu)\triangle\in.$

Since $\tau_{r(q)-1}(t/2\mu)\to 1$ as $q\to\infty$ and $\partial_{r(q)-1}(t/2\mu)\to 0$ as $q\to\infty$ for every t. It follows that

$$1-\in\ \geq H(b_{r(q)},b_{s(q)},t)>-\in,$$

and $$\in\ \leq E(b,b_{s(q)},t)>-\in.$$

Undoubtedly, this generates a contradiction. Hence $\{b_q\}$ is a Cauchy sequence in U. Since U is complete so we can find h∈U such that $\lim_{q\to\infty} b_q = h$.

Assume that, $h\neq Ah$, then there exists $t>0$ such that $H(h, Ah, t)\neq 1$ or $E(h, Ah, t)\neq 0$.

For this $t>0$, $H(Ab_q, Ah, kt)\geq H(b_q, Ab_q, t)\circ H(h, Ah, t)$, by contractive condition (3.1), $H(b_q+1, Ah, kt)\geq H(b_q, b_{q+1}, t)\circ H(h, Ah, t)$.

In limiting case as $t\to\infty$, $H(h, Ah, kt)\geq H(h, Ah, t)$.

As $H(h, Ah, t)\neq 1$, the above inequality yields a contradiction to the fact that $H(j, h, \cdot)$ is strictly increasing.

Moreover, $E(Ab_q, Ah, kt)\leq E(b_q, Ab_q, t)\triangle E(h, Ah, t)$, by contractive condition(3.2). That is $E(b_{q+1}, Ah, kt)\leq E(b_q, b_{q+1}, t)\triangle E(h, Ah, t)$.

In limiting case as $q\to\infty$, $E(h, Ah, kt)\leq E(h, Ah, t)$.

As $E(h, Ah, t)\neq 0$ the above inequality yields a contradiction to the fact that $E(b, h,\cdot)$ is strictly decreasing. Hence $h = Ah$.

For uniqueness, let d and f be two fixed points of A. So, $h = Ah$ and $f = Af$. Then

$$H(h, Ah, t) = 1,\ H(f, Af, t) = 1$$

and

$$E(h, Ah, t) = 0,\ E(f, Af, t) = 0 \text{ for all } t>0.$$

now,

$$1\geq H(d, f, t) = H(Ad, Af, t)\geq H(h, Ah, t/k)\circ H(f, Af, t/k)= 1,$$

$$0\leq E(h, f, t) = E(Ah, Af, t)\leq E(h, Ah, t/k)\triangle E(f, Af, t/k)= 0.$$

By the definition of IFCbMS, we have $f = h$.

Abbreviations used

BS- Banach space

FS- Fuzzy Set

FMS- Fuzzy metric space

IFMS- Intuitionistic fuzzy metric space

FCMS- Fuzzy cone metric space

FCbMS- Fuzzy cone-b metric space

IFbMS- Intuitionistic fuzzy-b metric space

IFCbMS- Intuitionistic fuzzy cone-b metric space

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